Efficient Helmholtz Equation Solver for Frequency Domain Waveform Inversion Based on the Decomposition into One-Way Wave Equations

Rehman Ali[1], Feiyu Wang[2], Trevor Mitcham[1], Nebojsa Duric[1]

[1]Department of Imaging Sciences, University of Rochester Medical Center

[2]Department of Mechanical Engineering, University of Rochester

UNIVERSITY of ROCHESTER MEDICAL CENTER

HAJIM SCHOOL OF ENGINEERING & APPLIED SCIENCES UNIVERSITY of ROCHESTER

MOTIVATION AND BACKGROUND

- Frequency-domain waveform inversion for ultrasound transmission tomography with a ring array transducer (Figure 1) often needs a computationally efficient solution to the Helmholtz equation.

Figure 1. Ring-Array Ultrasound Tomography System (SoftVue™) Developed by Delphinus Medical Technologies.

- Helmholtz equation:
  - $(\nabla^2 + \omega^2 s^2)\boldsymbol{u}(\omega, \boldsymbol{s}) = \boldsymbol{\delta}(\omega)$
    - pressure field $\boldsymbol{u}$ (over 2D space)
    - source excitation $\boldsymbol{\delta}$ (e.g,, point source in 2D)
    - slowness $\boldsymbol{s}$ (reciprocal of sound speed in 2D)
    - frequency $\omega = 2\pi f$
- In matrix-vector form:
  - $A(\omega, \boldsymbol{s})\boldsymbol{u}(\omega, \boldsymbol{s}) = \boldsymbol{\delta}(\omega)$
    - where $A(\omega, \boldsymbol{s})$ is discretization of Helmholtz operator $(\nabla^2 + \omega^2 s^2)$
    - we solve for $\boldsymbol{u}(\omega, \boldsymbol{s})$

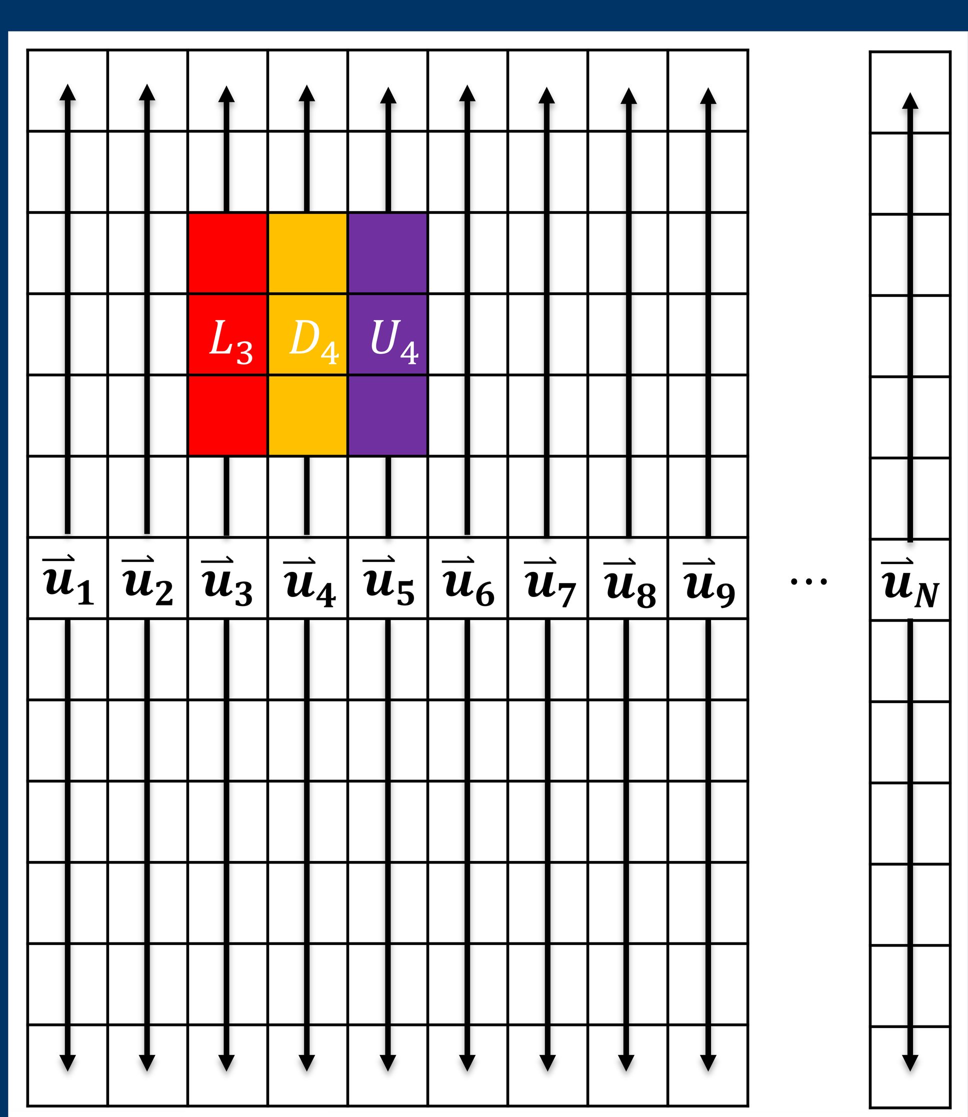

Figure 2. Visualization of the Computational Grid with the 9-Point Stencil Discretization. The full pressure field $\boldsymbol{u}$ is split into columns $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_N$ on the computational grid. The L_i, D_i, and U_i blocks of the block-tridiagonal A matrix result from the 9-point stencil ($L_i/D_i/U_i$ partitions shown).

BLOCK LU SOLUTION TO HELMHOLTZ EQUATION

- For a 9-point stencil (Figure 2), $A(\omega, \boldsymbol{s})$ is a block tridiagonal matrix of tridiagonal submatrices [1]:

$$A = \begin{bmatrix} D_1 & U_1 & & & \\ L_1 & D_2 & U_2 & & \\ & \ddots & \ddots & \ddots & \\ & & L_{N-2} & D_{N-1} & U_{N-1} \\ & & & L_{N-1} & D_N \end{bmatrix}$$

- Block LU decomposition of $A(\omega, \boldsymbol{s})$:

$$L = \begin{bmatrix} T_1 & & & \\ L_1 & T_2 & & \\ & \ddots & \ddots & \\ & & L_{N-2} & T_{N-1} & \\ & & & L_{N-1} & T_N \end{bmatrix}$$

$$U = \begin{bmatrix} I & T_1^{-1}U_1 & & & \\ & I & T_2^{-1}U_2 & & \\ & & \ddots & \ddots & \\ & & & I & T_{N-1}^{-1}U_{N-1} \\ & & & & I \end{bmatrix}$$

- Solve $A\boldsymbol{u} = \boldsymbol{\delta}$ for $\boldsymbol{u}$ using block LU decomposition
  - $A\boldsymbol{u} = (LU)\boldsymbol{u} = L(U\boldsymbol{u}) = \boldsymbol{\delta}$
  - Two steps (Figure 3): 1) $L\boldsymbol{v} = \boldsymbol{\delta}$, 2) $U\boldsymbol{u} = \boldsymbol{v}$
  - $\boldsymbol{u} = \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \\ \vdots \\ \vec{u}_N \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_N \end{bmatrix}$, $\boldsymbol{\delta} = \begin{bmatrix} \vec{\delta}_1 \\ \vec{\delta}_2 \\ \vdots \\ \vec{\delta}_N \end{bmatrix}$ (Figure 2)

  1) L-system is solved by a forward sweep: $\vec{v}_1 = T_1^{-1}\vec{\delta}_1$ and $\vec{v}_n = T_n^{-1}(\vec{\delta}_n - L_{n-1}\vec{v}_{n-1})$ for $n = 2, \ldots, N$.

  2) U-system is solved by a backward sweep: $\vec{u}_N = \vec{v}_N$ and $\vec{u}_n = \vec{v}_n - T_n^{-1}U_n\vec{u}_{n+1}$ for $n = 2, \ldots, N$.

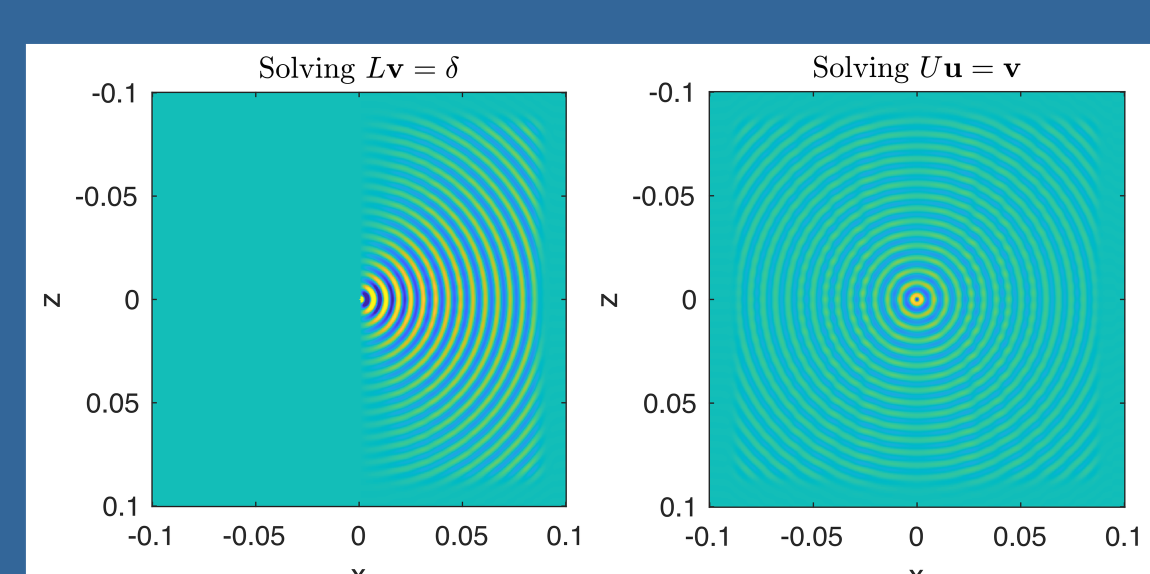

Figure 3. Visual demonstration of the Block LU Helmholtz Equation Solver: Forward sweep via $L\boldsymbol{v} = \boldsymbol{\delta}$ and backward sweep via $U\boldsymbol{u} = \boldsymbol{v}$

LU-LIKE DECOMPOSITION VIA ONE-WAY WAVE EQUATIONS

- Helmholtz equation: $(\nabla^2 + \omega^2 s^2)\boldsymbol{u}(\omega, \boldsymbol{s}) = \boldsymbol{\delta}(\omega)$
  - $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \omega^2 s^2\right)\boldsymbol{u} = \boldsymbol{\delta}$
  - $\left(\frac{\partial}{\partial x} + i\sqrt{\omega^2 s^2 + \frac{\partial^2}{\partial y^2}}\right)\left(\frac{\partial}{\partial x} - i\sqrt{\omega^2 s^2 + \frac{\partial^2}{\partial y^2}}\right)\boldsymbol{u} = \boldsymbol{\delta}$
- Equivalent to an LU-like decomposition in PDE form with the following one-way wave equations:
  - Forward sweep: $\left(\frac{\partial}{\partial x} + i\sqrt{\omega^2 s^2 + \frac{\partial^2}{\partial y^2}}\right)\boldsymbol{v} = \boldsymbol{\delta}$
  - Backward sweep: $\left(\frac{\partial}{\partial x} - i\sqrt{\omega^2 s^2 + \frac{\partial^2}{\partial y^2}}\right)\boldsymbol{u} = \boldsymbol{v}$
- Each one-way wave equation is implemented using either the **Fourier split-step method** [2] or **phase-shift-plus-interpolation (PSPI)** [3].
  - Mixed-domain (wavenumber & space) methods
    - FFTs between wavenumber & space
  - Highly parallelizable on the GPU
  - More memory efficient than block LU solver

WAVEFORM INVERSION ALGORITHM

- Full Waveform Inversion (FWI) Objective Function:
  - $E(\boldsymbol{s}; \omega) = \frac{1}{2}\sum_{i=1}^{N}\left\|\boldsymbol{p}_i(\omega, \boldsymbol{s}) - \boldsymbol{p}_{\boldsymbol{obs},i}(\omega)\right\|_2^2$
  - $\boldsymbol{p}_i$ and $\boldsymbol{p}_{\boldsymbol{obs},i}$ are the simulated and observed pressure measurements on the ring array for the i^{th} single-element transmission $\boldsymbol{\delta}_i$
  - $\boldsymbol{p}_i$ on sampled on ring array ($\boldsymbol{p}_i = K\boldsymbol{u}_i$) where K samples $\boldsymbol{u}_i$ at each element of ring array
  - Objective: find slowness profile $\boldsymbol{s}$ that causes simulation $\boldsymbol{p}_i$ to agree with observation $\boldsymbol{p}_{\boldsymbol{obs},i}$
  - Gradient image $\nabla_s E$ is equivalent to a CT-like backprojection of the error $\boldsymbol{p}_i - \boldsymbol{p}_{\boldsymbol{obs},i}$ (Figure 4).

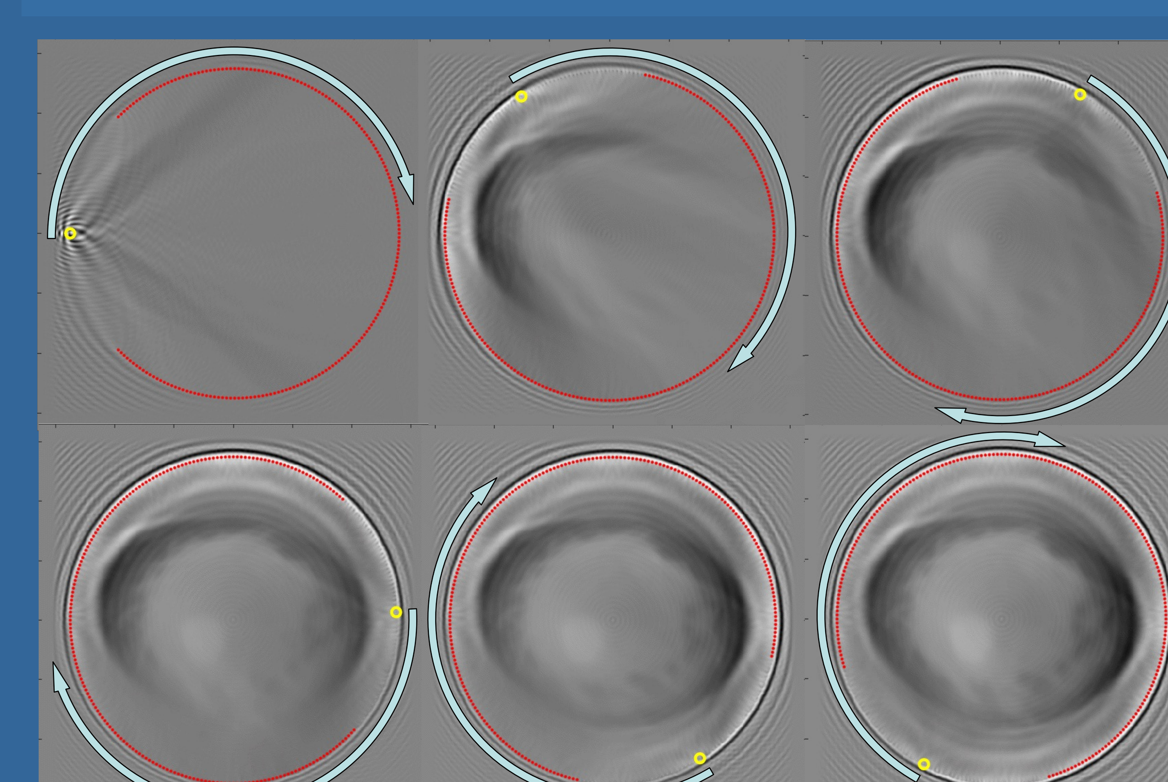

Figure 4. Visualization of $\nabla_s E$ the backprojection of the error $\boldsymbol{p}_i - \boldsymbol{p}_{\boldsymbol{obs},i}$ onto slowness profile during waveform inversion

VERIFICATION+COMPARISON TO BLOCK LU SOLUTION

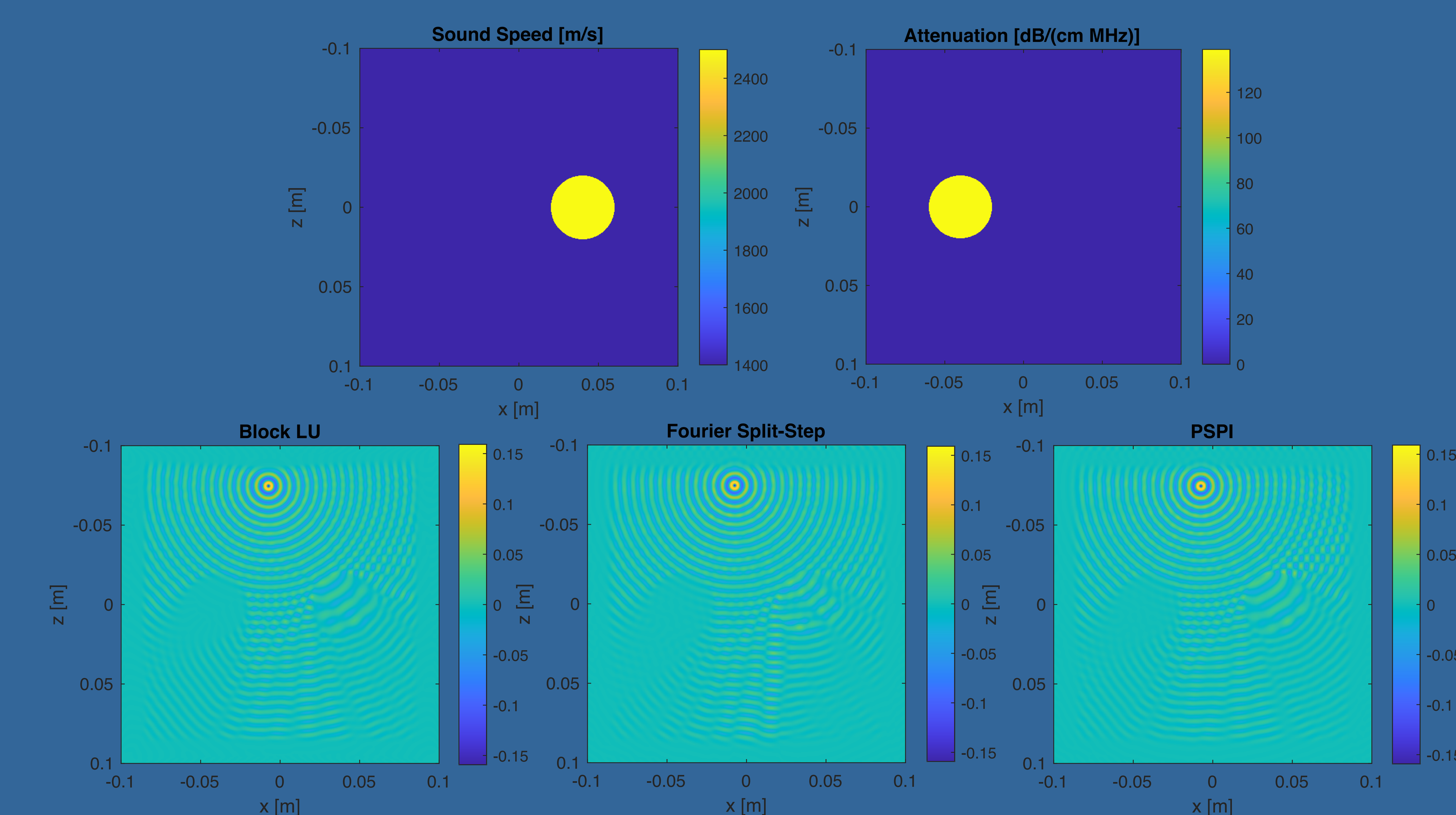

Figure 5. Comparison of One-Way Wave Equation Decomposition (via Fourier split-step method and PSPI) to the Block LU Decomposition for the Helmholtz Equation

IN-VIVO DEMO WITH WAVEFORM INVERSION

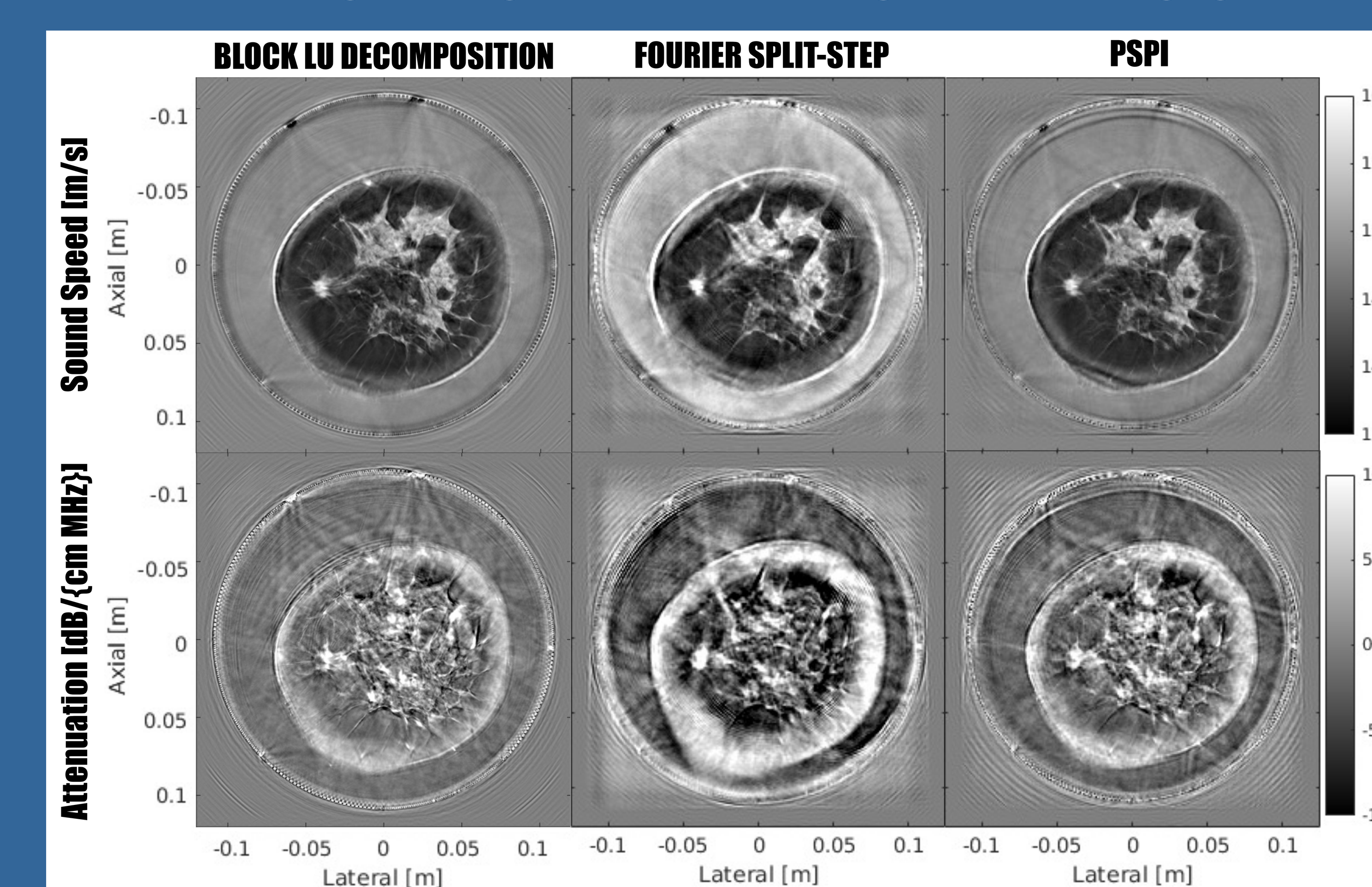

Figure 6. Comparison of FWI results with each Helmholtz equation solvers

CONCLUSIONS

- LU-like decomposition possible using one-way wave equations.
- PSPI is more accurate than the Fourier split-step method.

REFERENCES

1) Gander, M. J., & Zhang, H. (2019). A class of iterative solvers for the Helmholtz equation: Factorizations, sweeping preconditioners, source transfer, single layer potentials, polarized traces, and optimized Schwarz methods. Siam Review, 61(1), 3-76.
2) Wiskin, J. W., Borup, D. T., Iuanow, E., Klock, J., & Lenox, M. W. (2017). 3-D nonlinear acoustic inverse scattering: Algorithm and quantitative results. IEEE transactions on ultrasonics, ferroelectrics, and frequency control, 64(8), 1161-1174.
3) Gazdag, J., & Sguazzero, P. (1984). Migration of seismic data by phase shift plus interpolation. Geophysics, 49(2), 124-131.