

# Extending Retrospective Encoding for Robust Recovery of the Multistatic Data Set

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**Abstract**—Robust recovery of multistatic synthetic aperture data from conventional ultrasound sequences can enable complete transmit-and-receive focusing at all points in the field of view without the drawbacks of virtual-source synthetic aperture and further enables more advanced imaging applications, such as backscatter coherence, sound speed estimation, and phase aberration correction. Recovery of the multistatic data set has previously been demonstrated on a steered transmit sequence for phased arrays using an adjoint-based method. We introduce two methods to improve the accuracy of the multistatic data set. We first modify the original technique used for steered transmit sequences by applying a ramp filter to compensate for the nonuniform frequency scaling introduced by the adjoint-based method. Then, we present a regularized inversion technique that allows additional aperture specification and is intended to work for both steered transmit and walking aperture sequences. The ramp-filtered adjoint and regularized inversion techniques, respectively, improve the correlation of the recovered signal with the ground truth from 0.9404 to 0.9774 and 0.9894 in steered transmit sequences and 0.4610 to 0.4733 and 0.9936 in walking aperture sequences.

**Index Terms**—Frequency domain beamforming, multistatic synthetic aperture, retrospective encoding, signal recovery.

## I. INTRODUCTION

COMMERCIAL ultrasound systems generally rely on focused transmissions to interrogate human tissue. The backscattered echoes from each transmission are dynamic-receive focused into A-lines, which, together, composes the final ultrasound image. In this case, receive focusing predominantly dictates the resolution away from the transmit focal depth. In the absence of aberration, this generally means

that transmit focusing limits the resolution. In medical ultrasound, synthetic aperture refers to a set of imaging techniques that rely on the linear superposition of pulse-echo responses from multiple transmit events to improve transmit focusing in the B-mode image. Imaging applications that utilize synthetic aperture, such as coherence imaging [1], [2], sound speed estimation [3], [4], and phase-aberration correction [5], depend on transmit focusing.

The main advantage of the synthetic aperture is the flexibility with which transmit focusing can be synthesized without needing to retransmit ultrasonic pulses. Multistatic synthetic aperture uses receive channel data from single-element transmissions rather than from focused transmit beams. Multistatic channel data enable greater control of transmit focusing in the imaging process and enable complete transmit-and-receive focusing at all points in the ultrasound image [6]; however, single-element transmit sequences generally have low signal-to-noise ratio (SNR) and result in a slower frame rate.

As an alternative to single-element transmissions, Hadamard and S-sequences provide efficient spatial encoding and decoding schemes that can significantly boost the SNR of multistatic channel data with the same number of transmit events [7], [8]. One limitation of the Hadamard encoding is that it requires hardware that performs pulse inversion on an element-by-element basis. The S-sequence overcomes this limitation with a binary coding but sacrifices SNR because only half the total number of transmit elements are used on each transmit event. Delay-encoded synthetic transmit aperture (DE-STA) [9]–[11] tries to maintain the SNR boost of the Hadamard encoding while avoiding pulse inversion by using delays that give an approximately 180° phase shift between pulses. Temporal encoding and decoding [12]–[14] can also be used to improve the SNR of individual element transmissions, at the cost of harmonic imaging performance, transmit pulse sequence complexity, and possibly imaging frame rate.

Most commercial ultrasound scanners still utilize focused transmissions. Rather than isolating the single-element contributions to each transmit beam, they use virtual source synthetic focusing to coherently compound transmit beams along parallel receive lines [15], [16] and retrospectively improve transmit focusing at all depths. Although coherently compounding transmit beams in this way improves SNR, the improvement in transmit focusing is ultimately limited by the spacing and coverage of transmit beams

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through each imaging point. Dual-stage beamforming schemes can reduce the computational complexity of this method by using either parallel receive beams [17], [18] or a single focused beamline from each transmission [19], but result in reduced synthetic focusing accuracy and image quality.

Recent efforts have instead used conventional focused transmission sequences to encode and decode the multistatic data set. Li *et al.* [20] use a regularized singular value decomposition (RSVD) to decode the multistatic data set from a steered plane-wave sequence. Retrospective encoding for conventional ultrasound sequences (REFoCUS) [21]–[23] generalizes the recovery of the multistatic data set to focused, diverging, and plane-wave transmissions by providing a more general framework for transmissions with arbitrary delays [24]. The REFoCUS algorithm can be interpreted as an adjoint of the transmit beamforming process. In the time domain, REFoCUS consists of advancing the channel data from each transmit beam by the delays applied to each transmit element and summing across transmit beams to recover the signals from each transmit element in the multistatic data set. REFoCUS has previously been applied to a sequence of steered transmit beams from a phased-array and has been shown to yield equivalent imaging resolution as the multistatic data set [22]. However, adjoints do not generally serve as inversion operators and may limit the accuracy of the recovered multistatic data set. One goal of this work is to reexamine the performance and limitations of the adjoint-based approach applied to the steered transmit sequences. Analysis of the matrix formulation associated with steered transmit sequences reveals a ramp-filter correction to the original adjoint-based scheme, similar to its use in computed tomography (CT) [25], [26] and geophysical imaging [27].

We recently proposed a new technique for recovering the multistatic data set from a set of focused transmit beams [28], which begins by generalizing the transmit beamforming model to include both transmit delay and apodization when relating the single-element transmit responses to the channel data from focused transmit beams. This additional modeling of apodization, previously omitted by REFoCUS [22], is required for walking transmit apertures. When this transmit beamforming relationship is described in the frequency domain, the transfer function between the multistatic data set and focused-beam channel data can be written as a frequency-dependent matrix [9]–[11], [20]–[24]. In order to more accurately and robustly recover the multistatic data set, we use the Tikhonov-regularized inversion of this matrix, rather than the adjoint (or conjugate transpose), to perform the model inversion. This work aims to extend [28] by examining the performance of each multistatic data set recovery scheme on both steered and walking-aperture transmit sequences. Furthermore, this work provides an in-depth analysis of how each recovery scheme impacts the accuracy of the recovered multistatic channel data, the k-space response of each recovered single-element transmit, the coherence of focused channel data, and the achievable imaging resolution and contrast.

## II. THEORY OF MULTISTATIC DATA SET RECOVERY

### A. Forward Model of Transmit Beamforming

The channel data at receive element  $R \in [1, \dots, N_{\text{elem}}]$  due to a pulse emitted by transmit element  $T \in [1, \dots, N_{\text{elem}}]$  are denoted as  $u_{\text{TR}}(t)$ . Indexed over all possible  $N_{\text{elem}}N_{\text{elem}}$  transmit and receive element pairs,  $u_{\text{TR}}(t)$  is the complete multistatic synthetic aperture data set, without any delays or apodization applied. If a delay  $\tau_{nT}$  and an apodization  $w_{nT}$  are applied to each transmit element  $T$  to generate a (focused) transmit beam  $n \in [1, \dots, N_{\text{beams}}]$ , then the channel data received at element  $R$  due to transmit beam  $n$  can be written as

$$s_{nR}(t) = \sum_{T=1}^{N_{\text{elem}}} w_{nT} u_{\text{TR}}(t - \tau_{nT}). \quad (1)$$

In the frequency domain, this delay-and-sum is equivalent to

$$S_{nR}(f) = \sum_{T=1}^{N_{\text{elem}}} H_{nT}(f) U_{\text{TR}}(f) \quad (2)$$

where  $H_{nT}(f) = w_{nT} e^{-j2\pi f \tau_{nT}}$  and  $U_{\text{TR}}(f)$  is the Fourier transform of  $u_{\text{TR}}(t)$ . In the matrix form

$$\mathbf{S}_R(f) = \mathbf{H}(f) \mathbf{U}_R(f) \quad (3)$$

where  $\mathbf{S}_R$  and  $\mathbf{U}_R$  are column vectors of frequency-domain signals on receive element  $R$ , i.e.,  $\mathbf{S}_R(f) \in \mathbb{C}^{N_{\text{beams}}}$ ,  $\mathbf{U}_R(f) \in \mathbb{C}^{N_{\text{elem}}}$

$$\mathbf{S}_R(f) = [S_{1R}(f), \dots, S_{N_{\text{beams}}R}(f)]^T \quad (4)$$

$$\mathbf{U}_R(f) = [U_{1R}(f), \dots, U_{N_{\text{beams}}R}(f)]^T \quad (5)$$

and  $\mathbf{H}(f) \in \mathbb{C}^{N_{\text{beams}} \times N_{\text{elem}}}$ .

The goal of multistatic data set recovery is to obtain an accurate estimate  $\hat{\mathbf{U}}_R(f)$  of the multistatic data set  $\mathbf{U}_R(f)$  from the focused-transmit data set  $\mathbf{S}_R(f)$  at each frequency  $f$ . This can be accomplished by a pseudoinverse  $\mathbf{H}^\dagger(f)$  for  $\mathbf{H}(f)$  and applying it to  $\mathbf{S}_R(f)$  at each frequency  $f$

$$\hat{\mathbf{U}}_R(f) = \mathbf{H}^\dagger(f) \mathbf{S}_R(f) = \mathbf{H}^\dagger(f) \mathbf{H}(f) \mathbf{U}_R(f). \quad (6)$$

Ideally,  $\mathbf{H}^\dagger(f) \mathbf{H}(f)$  would equal the identity matrix  $\mathbf{I}$  and recovery of the multistatic data set  $\mathbf{U}_R(f)$  would be exact; thus, obtaining an accurate  $\hat{\mathbf{U}}_R(f)$  depends on achieving  $\mathbf{H}^\dagger(f) \mathbf{H}(f)$  as close to  $\mathbf{I}$  as possible. Three pseudoinverses  $\mathbf{H}^\dagger(f)$  are explored in the following.

### B. REFoCUS Method: Conjugate Transpose (Adjoint)

REFoCUS [22] was devised as a method to recover signals for single-element transmissions from a sequence of focused-transmit beams by applying the time advance-and-sum

$$\hat{u}_{\text{TR}}(t) = \sum_{n=1}^{N_{\text{beams}}} w_{nT} s_{nR}(t + \tau_{nT}). \quad (7)$$

In the frequency domain, this is equivalent to using the conjugate transpose, or adjoint, of  $\mathbf{H}(f)$  as the pseudoinverse

$$\mathbf{H}^\dagger(f) = \mathbf{H}^*(f). \quad (8)$$

REFoCUS was initially demonstrated on a steered transmit sequence for phased arrays with uniform apodization (i.e.,  $w_{nT} = 1$ ) so that  $H_{nT} = \exp(-j2\pi f \tau_{nT})$ . The goal of the following analysis is to develop a closed-form expression for the entries in  $\mathbf{H}^*(f)\mathbf{H}(f)$  and to identify potential improvements to the adjoint as a pseudoinverse.

For steered transmit beams on a phased array, the focal delays  $\tau_{nT}$ , parameterized by element location  $(x_T, 0)$ , steering angle  $\theta_n$ , and focal depth  $R$ , are given by

$$\tau_{nT} = \frac{1}{c} \left\{ R - \sqrt{x_T^2 - 2x_T R \sin \theta_n + R^2} \right\}. \quad (9)$$

The Taylor series expansion  $(x_T^2 - 2x_T R \sin \theta_n + R^2)^{1/2} \approx R - x_T \sin \theta_n + \mathcal{O}(x_T^2)$  and the small-angle approximation  $\sin \theta_n \approx \theta_n$  simplify  $H_{nT}$ , with uniform apodization, to

$$H_{nT} \approx \exp \left( -j \frac{2\pi f}{c} x_T \theta_n \right). \quad (10)$$

The  $a$ th row and  $b$ th column of  $\mathbf{H}^*(f)\mathbf{H}(f)$  can be calculated from (10) with a uniform angular spacing between beams such that  $\theta_n = n\Delta\theta - (1)/(2)(N_{\text{beams}} + 1)\Delta\theta$

$$\begin{aligned} (\mathbf{H}^*\mathbf{H})_{ab} &= \sum_{n=1}^{N_{\text{beams}}} H_{na}^* H_{nb} \\ &\approx e^{j\hat{\omega}_{ab} \left( \frac{N_{\text{beams}}+1}{2} \right)} \sum_{n=1}^{N_{\text{beams}}} \exp(-j\hat{\omega}_{ab} n) \\ &\approx \frac{\sin(\hat{\omega}_{ab} N_{\text{beams}}/2)}{\sin(\hat{\omega}_{ab}/2)} \end{aligned} \quad (11)$$

where  $\hat{\omega}_{ab} = (2\pi f)/(c)(x_b - x_a)\Delta\theta$ . Since  $(\mathbf{H}^*\mathbf{H})_{ab}$  corresponds to the contribution of transmit element  $b$  from the true multistatic data set to transmit element  $a$  in the recovered multistatic data set, (11) reveals crosstalk between elements when using the adjoint as the decoding scheme. Ideally, we would like  $\mathbf{H}^*(f)$  to satisfy  $\mathbf{H}^*(f)\mathbf{H}(f) = \mathbf{I}$  or, equivalently,  $(\mathbf{H}^*\mathbf{H})_{ab} = \delta_{ab}$ , where  $\delta_{ab}$  is the Kronecker delta. If  $\Delta x$  is the interelement pitch of the transducer array, and  $l = b - a$  is the offset between elements  $a$  and  $b$  on the array, then  $x_b - x_a = l\Delta x$  simplifies (11) to

$$(\mathbf{H}^*\mathbf{H})_{a,a+l} \approx N_{\text{beams}} \text{sinc} \left( \left( \frac{N_{\text{beams}} f \Delta\theta \Delta x}{c} \right) l \right). \quad (12)$$

If  $\mathbf{H}^*(f)\mathbf{H}(f)$  is approximated to be a circulant matrix, the discrete Fourier transform (DFT) matrix  $\mathcal{F}$  may be used as a change of basis to diagonalize  $\mathbf{H}^*(f)\mathbf{H}(f)$ . As a result, the diagonal entries of  $D[m] = (\mathcal{F}^* \mathbf{H}^*(f)\mathbf{H}(f) \mathcal{F})_{mm}$  are approximately the DFT of (12) with respect to  $l$

$$D[m] \approx \frac{c}{f \Delta\theta \Delta x} \text{rect} \left( \frac{c}{N_{\text{beams}} f \Delta\theta \Delta x} \left( \frac{m}{N_{\text{elem}}} \right) \right). \quad (13)$$

After applying the transformation  $k_x = (m)/(N_{\text{elem}} \Delta x)$ , the resulting transfer function summarizes how content at  $k_x$  and  $f$  in the multistatic data set get transformed by  $\mathbf{H}^*(f)\mathbf{H}(f)$

$$D(k_x) \approx G_{\text{passband}}(f) \text{rect} \left( \frac{k_x}{2k_{x,\text{max}}(f)} \right) \quad (14)$$

where  $G_{\text{passband}}(f) = (c)/(f \Delta x \Delta\theta)$  and  $k_{x,\text{max}}(f) = (N_{\text{beams}} f \Delta\theta)/(2c)$  are the passband gain and cutoff frequency, respectively.

### C. Ramp-Filtered Adjoint (for Phased-Array Sequence)

According to (14),  $\mathbf{H}^*(f)\mathbf{H}(f)$  will apply a gain of  $G_{\text{passband}}(f)$  across all lateral wavenumbers that fall within the passband given by  $|k_x| < k_{x,\text{max}}(f)$ , and any content that falls outside the passband is removed. Thus, using the adjoint as a pseudoinverse introduces two key problems: first, the passband gain should be unity across all frequencies (i.e.,  $G_{\text{passband}}(f) = 1$ ) in order for the adjoint to accurately approximate inversion, and second, any content that falls outside the passband is not recovered. The first problem can be resolved by using the reciprocal of  $G_{\text{passband}}(f)$  as a scaling factor to achieve a net-unity gain. Because the reciprocal of  $G_{\text{passband}}(f)$  is proportional to  $f$ , this scaling factor essentially acts as a ramp filter. Based on the derivation of (14), the following ramp-filtered adjoint may serve as a more accurate pseudoinverse for steered-transmit sequences on phased arrays:

$$\mathbf{H}^\dagger(f) = \frac{f \Delta x \Delta\theta}{c} \mathbf{H}^*(f) \quad (15)$$

where  $\Delta x$  is the transducer pitch and  $\Delta\theta$  is the angular spacing between transmit beams. The time domain equivalent is

$$\hat{u}_{\text{TR}}(t) = h_{\text{ramp}}(t) * \sum_{n=1}^{N_{\text{beams}}} s_{nR}(t + \tau_{nT}) \quad (16)$$

where the ramp filter  $h_{\text{ramp}}(t)$  given in discrete time index  $k$  with sampling rate  $f_s$  is

$$h_{\text{ramp}} \left( \frac{k}{f_s} \right) = \frac{f_s \Delta x \Delta\theta}{c} \left[ \frac{1}{2} \text{sinc}(k) - \frac{1}{4} (\text{sinc}(k/2))^2 \right]. \quad (17)$$

For imaging, the scaling of channel data is arbitrary, so the factor of  $\Delta x \Delta\theta/c$  may be ignored and (15) may be simplified to  $\mathbf{H}^\dagger(f) = f \mathbf{H}^*(f)$  in practice. This ramp-filter correction should be limited to the passband of the transducer to avoid amplifying high-frequency components of noise. This ramp-filter correction is common in CT, where radial sampling in  $k$ -space at multiple angles produces redundant sampling at the center of  $k$ -space [25], [26], and some geophysical applications also use frequency scaling to better model inversion using adjoint operators as well [27].

### D. Tikhonov-Regularized Inversion

Walking aperture transmit sequences on linear arrays raise additional complications not easily addressed by simple corrections to the adjoint. The delays for the walking aperture transmit sequence with focal depth  $z_{\text{foc}}$  are

$$\tau_{nT} = \frac{1}{c} \left\{ \sqrt{x_n^2 + R^2} - \sqrt{(x_T - x_n)^2 + z_{\text{foc}}^2} \right\} \quad (18)$$

for element locations within the active subaperture

$$w_{nT} = \begin{cases} 1, & \text{for } |x_T - x_n| \leq \frac{D}{2} \\ 0, & \text{otherwise} \end{cases} \quad (19)$$

TABLE I  
FIELD II SIMULATION SETTINGS

| Parameter            | Value  | Units    |
|----------------------|--------|----------|
| Array Geometry       | Linear | -        |
| Number of Elements   | 96     | elements |
| Element Pitch        | 0.154  | mm       |
| Kerf                 | 0.014  | mm       |
| Center Frequency     | 5      | MHz      |
| Fractional Bandwidth | 0.7    | -        |
| Speed of Sound       | 1540   | m/s      |
| Sampling Frequency   | 100    | MHz      |

where  $(x_n, 0)$  are the beam origin locations and  $D$  is the maximum subaperture width. Rather than recovering an estimate of the multistatic data set by applying a modified adjoint operator, the Tikhonov-regularized inversion attempts to find the multistatic data set  $\hat{U}_R(f)$  that minimizes

$$\|H(f)\hat{U}_R(f) - S_R(f)\|_2^2 + (\gamma \|H(f)\|_2 \|\hat{U}_R(f)\|_2)^2 \quad (20)$$

where  $\|H(f)\|_2 = \sigma_{\max}(f)$  is the maximum singular value of  $H(f)$  and  $\gamma$  is a regularization parameter that penalizes  $\|\hat{U}_R(f)\|_2$ . The pseudoinverse that results from minimizing the objective function (20) with respect to  $\hat{U}_R(f)$  is

$$H^\dagger(f) = [H^*(f)H(f) + (\gamma \sigma_{\max}(f))^2 I]^{-1} H^*(f). \quad (21)$$

There is no obvious time-domain equivalent for this pseudoinverse. A similar regularization approach is combined with the singular value decomposition (SVD) to decompose a plane-wave sequence into single-element contributions. Our work directly performs regularized inversion without requiring an initial SVD step. Our approach also includes transmit apodization, which was not previously incorporated in REFoCUS [22], to accommodate the walking-aperture transmit sequence.

### III. METHODS

#### A. Transmit Simulation

The complete multistatic data set was simulated in Field II [29], [30] for a medium of random diffuse point scatterers whose reflectivities were modified to simulate an anechoic lesion and two hyperechoic lesions, each with a radius of 2.25 mm. The anechoic lesion is centered at 30 mm, and the hyperechoic lesions are centered at 10- and 20-mm depth with +6- and +12-dB contrast, respectively, relative to the background. Simulation parameters are described in Table I.

Focused transmit beams were created by delaying receive channel data from each transmit element and summing to obtain focused transmit data based on (1). Table II describes the phased-array transmit sequence, and Table III describes the walking aperture transmit sequence, both of them are created using the same multistatic data set.

An additional Field II simulation corresponding to the L12-3v probe and the walking aperture sequence shown in Table IV was used to assess the recoverability of the multistatic data set from this walking aperture transmit sequence.

TABLE II  
PHASED-ARRAY TRANSMIT SEQUENCE IN FIELD II

| Parameter                  | Value       | Units   |
|----------------------------|-------------|---------|
| Focal Depth                | 30          | mm      |
| F-number                   | 3.75        | -       |
| Steering Angle Range       | [-45, 45]   | degrees |
| Steering Angle Step        | 0.5         | degrees |
| Total Number of Beams      | 181         | -       |
| Transmit Elements Per Beam | 96          | -       |
| Transmit Apodization       | Rectangular | -       |

TABLE III  
WALKING-APERTURE TRANSMIT SEQUENCE IN FIELD II

| Parameter                           | Value         | Units   |
|-------------------------------------|---------------|---------|
| Focal Depth                         | 30            | mm      |
| Transmit Angle                      | 0             | Degrees |
| Beam Origin Range                   | [-3.65, 3.65] | mm      |
| Beam Origin Step                    | 0.05          | mm      |
| Total Number of Beams               | 147           | -       |
| Aperture Width (Around Beam Origin) | 8             | mm      |
| F-number                            | 3.75          | -       |
| Transmit Elements Per Beam          | 50-52         | -       |
| Transmit Apodization                | Rectangular   | -       |

TABLE IV  
WALKING-APERTURE TRANSMIT SEQUENCE ON VERASONICS L12-3V

| Parameter                           | Value         | Units    |
|-------------------------------------|---------------|----------|
| Array Geometry                      | Linear        | -        |
| Number of Elements                  | 192           | elements |
| Element Pitch                       | 0.2           | mm       |
| Center Frequency                    | 7.8           | MHz      |
| Focal Depth                         | 41.4          | mm       |
| Transmit Angle                      | 0             | Degrees  |
| Beam Origin Range                   | [-19.1, 19.1] | mm       |
| Beam Origin Step                    | 1             | element  |
| Aperture Width (Around Beam Origin) | 128           | elements |
| Transmit Apodization                | Rectangular   | -        |

#### B. Experimental Data Acquisition

Table IV summarizes the walking-aperture focused-transmit sequence implemented on the Verasonics Vantage 256 scanner for the L12-3v probe. As the beam origin is walked to the edges of the array, the number of active elements in each beam decreases to 65 elements. Radio frequency channel data were acquired from an ATS Model 549 model phantom (1460 m/s) and an obese Zucker rat (Charles River Laboratories Inc., Wilmington, MA, USA) under a Stanford-approved IACUC protocol (APLAC-32987). These data sets were used to experimentally test each pseudoinverse for multistatic data set recovery. The performance of multistatic data set recovery was measured by resolution and spatial coherence in the phantom experiments.

#### C. Analysis of $H^\dagger(f)H(f)$

1) *Singular Value Decomposition*: The SVD of  $H(f)$  is

$$H(f) = U(f)\Sigma(f)V^*(f) \quad (22)$$

where  $\mathbf{U}(f) \in \mathbb{C}^{N_{\text{beams}} \times N_{\text{beams}}}$ ,  $\mathbf{\Sigma}(f) \in \mathbb{R}^{N_{\text{beams}} \times N_{\text{elem}}}$ , and  $\mathbf{V}(f) \in \mathbb{C}^{N_{\text{elem}} \times N_{\text{elem}}}$ . The columns of  $\mathbf{U}(f)$  are singular vectors across transmit beams, and the columns of  $\mathbf{V}(f)$  are singular vectors across transmit elements. For the purposes of recovering the multistatic data set, the singular vectors in  $\mathbf{V}(f)$  are the orthogonal modes along which encoding and decoding occur. The diagonal entries of  $\mathbf{\Sigma}(f)$  are singular values for each singular vector. The distribution of singular values provides information about the relative sensitivity of inversion to each orthogonal mode. For each pseudoinverse,  $\mathbf{H}^\dagger(f)\mathbf{H}(f)$  can be written in terms of  $\mathbf{V}(f)$  and  $\mathbf{\Sigma}(f)$ . The SVD was used to analyze the information content encoded by the walking-aperture and phased-array transmit sequences.

**2) Lateral Wavenumber Response:** The  $\mathbf{H}^\dagger(f)\mathbf{H}(f)$  matrix summarizes the encoding-and-decoding process that occurs at each temporal frequency. The transmit sequence acts as an encoding process, while the choice of pseudoinverse acts as the decoding process. The lateral wavenumber, or  $k_x$ , response of  $\mathbf{H}^\dagger(f)\mathbf{H}(f)$  provides information about how lateral wavenumbers on the transducer array are transformed by the complete encoding-and-decoding process at each temporal frequency. The  $k_x$  response of  $\mathbf{H}^\dagger(f)\mathbf{H}(f)$  can be calculated using the DFT matrix  $\mathcal{F}$  as a change of basis

$$D[m] = (\mathcal{F}^* \mathbf{H}^\dagger(f) \mathbf{H}(f) \mathcal{F})_{mm} \quad (23)$$

where  $(\mathcal{F}^* \mathbf{H}^\dagger(f) \mathbf{H}(f) \mathcal{F})_{mm}$  is the  $m$ th diagonal entry of  $\mathcal{F}^* \mathbf{H}^\dagger(f) \mathbf{H}(f) \mathcal{F}$ . The transfer function  $D(k_x)$  in  $k_x$ -space can be computed by transforming  $m$  to  $k_x$  as  $k_x = (m)/(N_{\text{elem}} \Delta x)$ . When using the adjoint as the pseudoinverse for the steered-transmit sequence,  $D(k_x)$  shows the need for a ramp filter [see (14)].

**3) K-Space for Transmit Wavefield:** The mathematical k-space representation of the transmit wavefield is shown in the Appendix. Transmit k-space recovery can be written as

$$\hat{\Phi}_{\text{recov},T}(\vec{k}, t) = \sum_{n=0}^{N_{\text{beams}}-1} H_{Tn}^\dagger(c|\vec{k}|) \hat{\Phi}_n(\vec{k}, t) \quad (24)$$

where  $\hat{\Phi}_{\text{recov},T}(\vec{k}, t)$  and  $\hat{\Phi}_n(\vec{k}, t)$  are the transmit k-spaces for recovered element  $T$  and transmit beam  $n$ , and  $H_{Tn}^\dagger(f)$  is one of the three pseudoinverses in (8), (15), or (21). The magnitude of the true and recovered k-space responses of the single-element transmit wavefield are compared to find the pseudoinverse that best recovers the true k-space response  $\hat{\Phi}_T(\vec{k}, t)$  of element  $T$ .

#### D. Measurement of Time-Domain Signal Recovery

In order to evaluate the accuracy of multistatic data set recovery, the correlation was measured between the recovered multistatic data set and the true multistatic data set directly simulated in Field II.

The amplitude of the time domain signal decreases with time due to the geometrical spreading of the acoustic wave. The time-gain compensation is applied to both the recovered and true multistatic data sets in order to adjust for this geometric spreading prior to measuring the signal correlation.

The correlation between the time-gain compensated data sets is

$$\rho = \frac{\text{COV}[\hat{u}_{\text{TR}}, u_{\text{TR}}]}{\sqrt{\text{COV}[\hat{u}_{\text{TR}}, \hat{u}_{\text{TR}}] \cdot \text{COV}[u_{\text{TR}}, u_{\text{TR}}]}} \quad (25)$$

where

$$\text{COV}[\hat{u}_{\text{TR}}, u_{\text{TR}}] = \int_{-\infty}^{\infty} t^2 \left( \sum_{T,R=1}^{N_{\text{elem}}} \hat{u}_{\text{TR}}(t) u_{\text{TR}}(t) \right) dt. \quad (26)$$

#### E. Performance Metrics

The performance of each pseudoinverse is measured by the lateral point-target resolution after image reconstruction and the spatial coherence of the recovered channel signals [31], [32]. Multistatic data are first processed by applying geometric delays across transmit and receive elements for each point in the final image. The resulting data are summed across the transmit element dimension leaving behind a stack of transmit-focused I/Q data for each element on the receive aperture.

Denote  $I_{x,z}[r]$  as these I/Q data for receive element  $r$  at location  $(x, z)$ . Spatial coherence given as function of element offset, or spatial lag ( $l$ ), is written as

$$R_{x,z}[l] = \frac{\sum_{i=1}^{N_{\text{elem}}-l} (I_{x,z}[i+l])^* (I_{x,z}[i])}{\sqrt{\left( \sum_{i=1}^{N_{\text{elem}}-l} |I_{x,z}[i+l]|^2 \right) \left( \sum_{i=1}^{N_{\text{elem}}-l} |I_{x,z}[i]|^2 \right)}}. \quad (27)$$

Spatial coherence is averaged within a region-of-interest containing speckle. In the Field II simulation, spatial coherence is averaged within each of the two hyperechoic lesions. In the phantom experiment, spatial coherence is averaged from a depth of 20 to 40 mm in an area covering half the width of the L12-3v transducer (−10–10 mm laterally). Lateral resolution is measured using the full-width at half-maximum (FWHM) of a single point target. In Field II, the point target is placed at a depth of 25 mm. In the phantom experiment, the point target is located at a depth of 20 mm.

### IV. RESULTS

#### A. Wavenumber Response and SVD Analysis of $\mathbf{H}(f)$

The  $\mathbf{H}^*(f)\mathbf{H}(f)$  matrix for the phased-array, or steered, transmit sequence described in Table II, is calculated based on the matrix description in (2) and delays given in (9). Fig. 1 compares the true  $\mathbf{H}^*(f)\mathbf{H}(f)$  matrix to its analytical approximation given by (12) for two frequencies. The agreement between the true  $\mathbf{H}^*(f)\mathbf{H}(f)$  and its approximation suggests that their SVD and wavenumber responses should also agree. This is confirmed by the results shown in Figs. 2 and 3.

Fig. 2 shows the  $k_x$  response of  $\mathbf{H}^*(f)\mathbf{H}(f)$  and its sinc approximation (12) for a range of frequencies. At each frequency, the  $k_x$  response of  $\mathbf{H}^*(f)\mathbf{H}(f)$  generally agrees with the  $k_x$  response of the sinc approximation. Each  $k_x$  response shows a roughly flat region, which we refer to as a passband, followed by a sharp drop-off. The passband gain, or the  $k_x$  response within the passband, appears to be

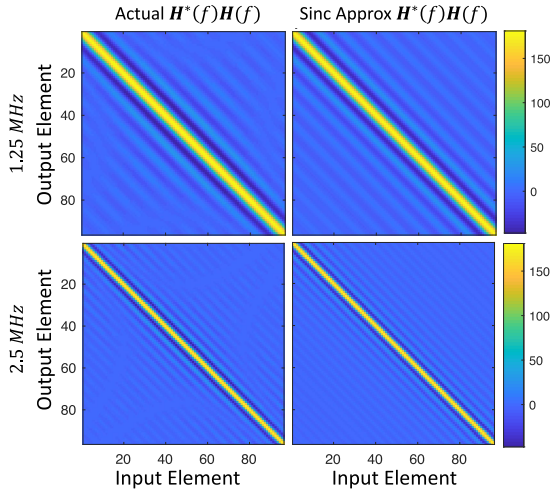


Fig. 1. Comparison of true  $H^*(f)H(f)$  to analytical sinc approximation at two different frequencies for the phased-array sequence shown in Table II. Input elements refer to elements in the true multistatic data set, and output elements refer to elements in the recovered multistatic data set.

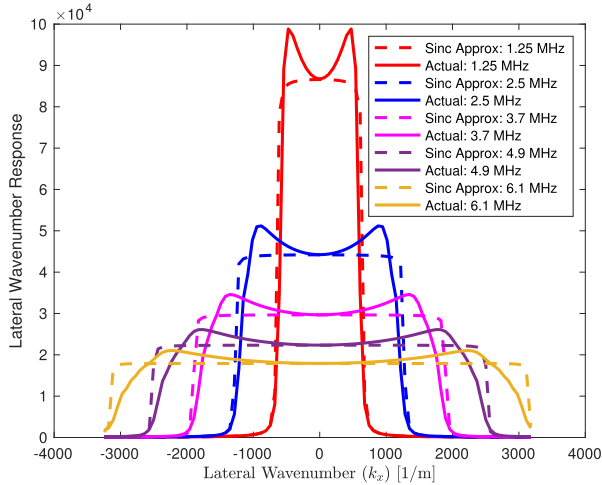


Fig. 2. Comparison between the wavenumber response of the true  $H^*(f)H(f)$  and analytical sinc approximation for the phased-array sequence shown in Table II.

inversely proportional to frequency. The cutoff wavenumber is proportional to the frequency.

Fig. 3 shows the squared singular values of  $H(f)$  in ascending order. At each frequency, most of the nonzero singular values contained in  $H(f)$  are captured by a small subset of components. The number of singular values within this flat “passband” is roughly proportional to the frequency, much like the wavenumber cutoff in the  $k_x$  response (see Fig. 2). Furthermore, the singular values within this passband are inversely proportional to frequency, much like the passband gain in the  $k_x$  response.

Fig. 4 shows the squared singular values of  $H(f)$ , in ascending order, for the walking aperture transmit sequence shown in Table III. The width and height of the distribution of singular values seem to be proportional and inversely proportional to frequency, respectively. Although these trends are similar to what is observed in Fig. 3, the singular values

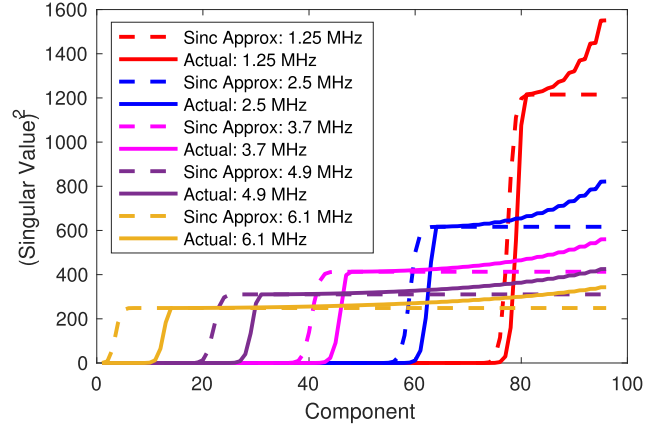


Fig. 3. Agreement between the singular values of the true  $H(f)$  and singular values based on the analytical sinc approximation of  $H^*(f)H(f)$  for the phased-array sequence shown in Table II.

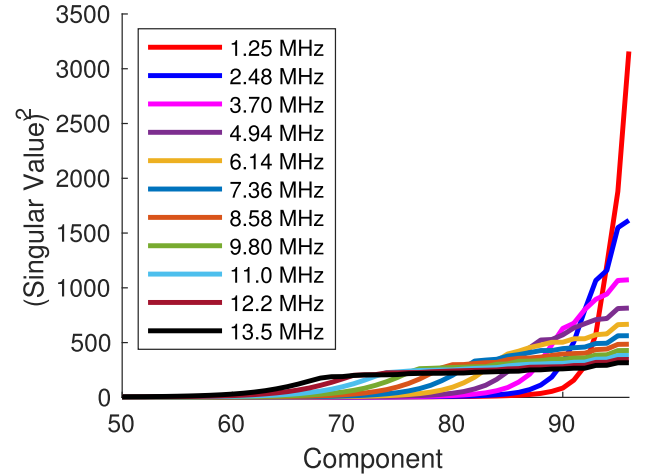


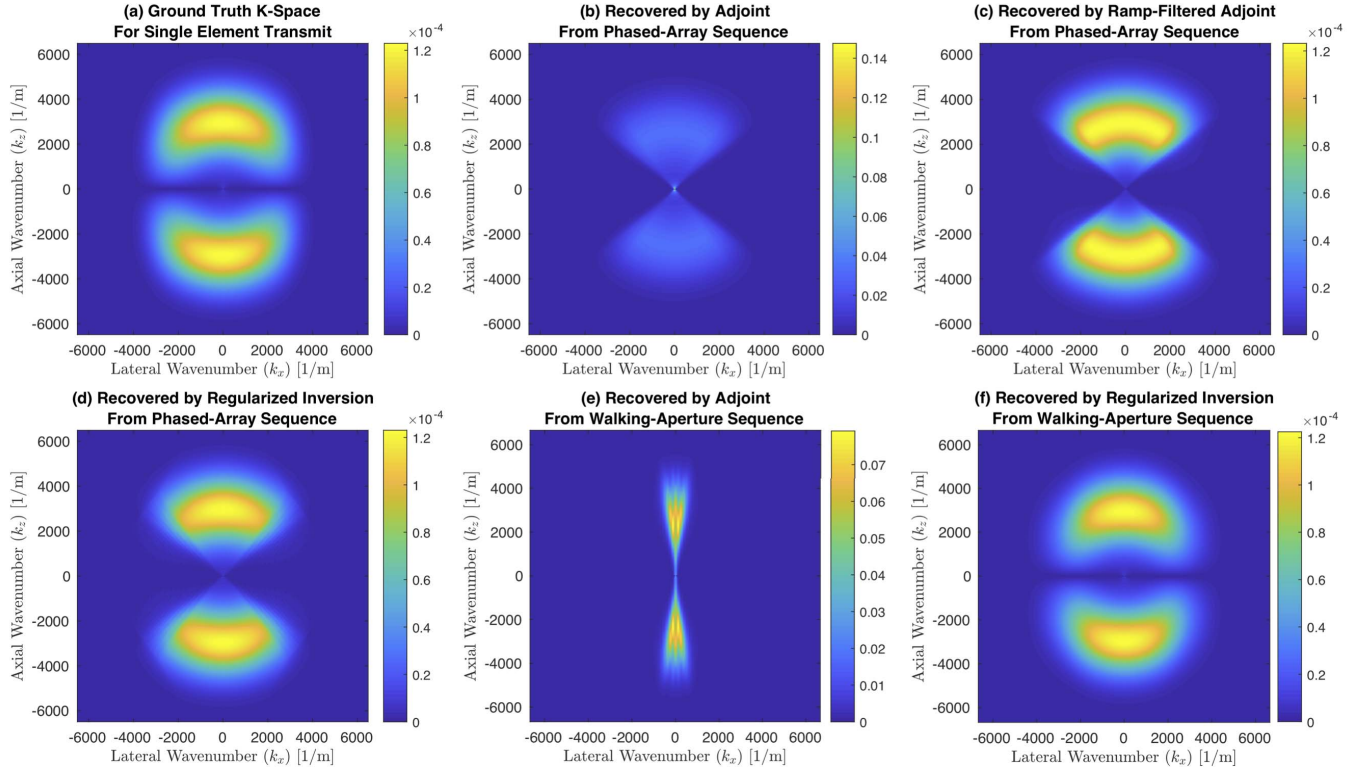
Fig. 4. Nonuniformity of the singular values of  $H(f)$  for the walking-aperture transmit sequence shown in Table III.

given in Fig. 4 vary more gradually so that the cutoff point and passband gain are no longer clearly identifiable.

### B. Recovery of the Single-Element Transmit K-Space

Fig. 5(a) shows the k-space for the wavefield from a single-element transmission as modeled by (30) and (31). The goal of each pseudoinverse is to recover the k-space of the single-element transmit wavefield from the k-spaces of each beam in the transmit sequence using (24).

Fig. 5(b)–(d) show the single-element k-space recovered from a phased-array sequence using the adjoint [Fig. 5(b)], the ramp-filtered adjoint [Fig. 5(c)], and the Tikhonov-regularized inversion [Fig. 5(d)] as pseudoinverses. As shown previously, the adjoint introduces a  $1/f$  scaling, which requires a ramp filter for correction. The temporal frequency corresponds to the radius  $f/c$  in k-space according to the dispersion relation  $(k_x^2 + k_z^2)^{1/2} = f/c$ . As a result, the ramp filter results in a radial weighting of k-space similar to CT. Both the ramp-filtered adjoint and the regularized inversion result in a k-space recovery that is limited to the transmission angles in the phased array sequence.



**Fig. 5.** (a) K-space magnitude for the wavefield of a single-element transmission. The goal of each pseudoinverse is to recover this k-space response from a sequence of transmit beams according to (24). (b)–(d) Recovery of single-element transmit k-space from phased-array sequence shown in Table II using (b) adjoint, (c) ramp-filtered adjoint, and (d) regularized inversion. The ramp filter eliminates the low-frequency bias in k-space, and the regularized inversion ( $\gamma = 0.1$ ) better models the element directivity in k-space. (e) and (f) Recovery of single-element transmit k-space from the walking-aperture sequence shown in Table III using (e) adjoint and (f) regularized inversion. The adjoint recovers only a limited range of angles in k-space, while regularized inversion ( $\gamma = 0.001$ ) recovers the full angular coverage and k-space magnitude of the single-element transmit wavefield.

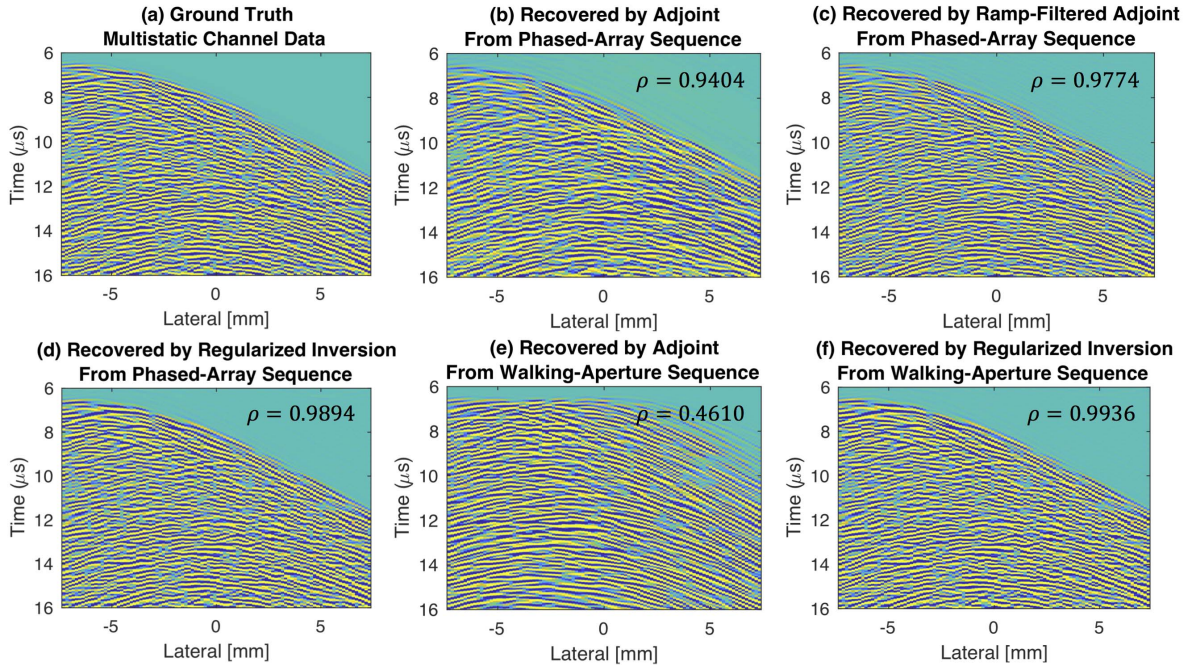
Although both pseudoinverses recover the k-space between  $-45^\circ$  and  $45^\circ$ , there is a difference in the recovery as a function of angle. The ramp-filtered adjoint produces a response that appears roughly constant as a function of angle. The regularized inversion produces a k-space response that most accurately shows the  $\cos\theta$  dipole directivity visible in the single-element k-space [see Fig. 5(a)]. This discrepancy in k-space recovery between the ramp-filtered adjoint and the Tikhonov-regularized inversion can be explained by the lateral wavenumber response shown in Fig. 2. Note that the passband of the lateral wavenumber response of the true  $\mathbf{H}^*(f)\mathbf{H}(f)$  is not perfectly flat. The wavenumber response rises at the edges of the passband. This cancels the  $\cos\theta$  directivity resulting in a flat k-space response as a function of angle when the ramp-filtered adjoint is used as the pseudoinverse.

Fig. 5(e) and (f) show the adjoint [Fig. 5(e)] and regularized inversion [Fig. 5(f)] recovery for the walking-aperture sequence. The adjoint can only recover a narrow range of angles in the k-space response of the single-element transmission and, ultimately, cannot reproduce the k-space for the single-element transmission [see Fig. 5(a)]. However, regularized inversion recovers the full angular coverage and k-space magnitude of the single-element transmit wavefield. Unlike with the phased-array sequence [see Fig. 5(d)], regularized inversion recovers the full angular coverage of the single element transmission from the walking aperture sequence [see Fig. 5(f)].

### C. Comparison of Pseudoinverses for Data Set Recovery

Fig. 6(a) shows the simulated receive channel data due to a transmission from the first element on the array. The goal of each pseudoinverse is to recover the multistatic data set from receive channel data collected from each beam in the transmit sequence. Fig. 6(b)–(d) shows the channel data recovered by each pseudoinverse for the phased-array transmit sequence. The channel data recovered by the adjoint [see Fig. 6(b)] have a 0.9404 correlation with the true multistatic data set. Differences in signal amplitudes are due to the low-frequency bias previously shown through  $k_x$  responses (see Fig. 2) and k-space [see Fig. 5(b)]. Ramp filtering [see Fig. 6(c)] removes these artifacts in the recovered signals and improves the correlation to 0.9774. Regularized inversion [see Fig. 6(d)] further improves signal correlation to 0.9894.

Fig. 6(e) and (f) show the channel data recovered using the adjoint and regularized inversion for the walking-aperture sequence. The adjoint [see Fig. 6(e)] results in only a 0.4610 correlation with the ground-truth multistatic data set. Because the ramp filter was designed as a correction to the adjoint for a phased-array sequence, it produces little improvement in the walking-aperture sequence. However, regularized inversion [see Fig. 6(f)] results in a highly accurate channel data recovery and achieves a correlation of 0.9936 with the true multistatic data set. These simulations show that regularized inversion can be more reliable than the adjoint for walking apertures.

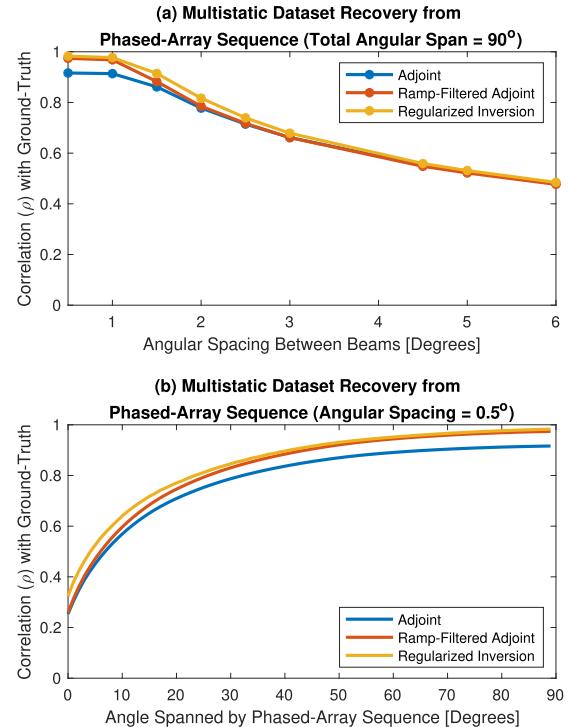


**Fig. 6.** Reconstruction of the multistatic data set from the phased-array and walking-aperture transmit sequences shown in Tables II and III. Images show the time-gain compensated received channel data versus time for the first transmit element on the array. (a) Ground-truth multistatic channel data; multistatic channel data recovered from phased-array sequence shown in Table II using (b) adjoint, (c) ramp-filtered adjoint, and (d) Tikhonov regularization with  $\gamma = 0.1$ ; multistatic channel data recovered from walking aperture sequence shown in Table III using (e) adjoint and (f) Tikhonov regularization with  $\gamma = 0.01$ . Correlation  $\rho$  between the true and recovered multistatic channel data is shown.

Figs. 7 and 8 show the effect of the extent and spacing of beams in each transmit sequence on the correlation between the multistatic data set recovered by each pseudoinverse and the true multistatic data set. Fig. 7 shows the effect of the angular extent and angular spacing of the beam in the phased-array transmit sequence. Fig. 8 shows the effect of the lateral extent and the spacing between beam origins in the walking-aperture transmit sequence. In each case, data set recovery with each pseudoinverse gradually improves with extent and gradually worsens as the spacing between beams increases.

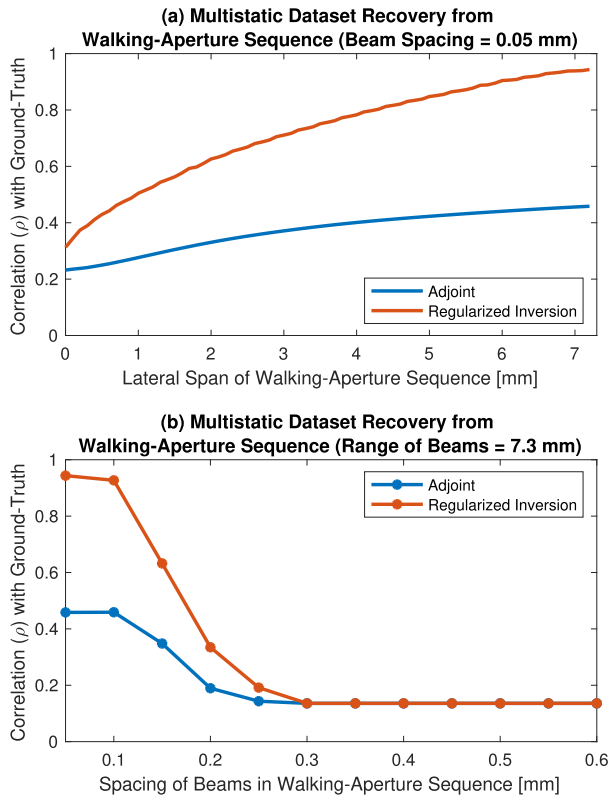
#### D. Multistatic Synthetic Aperture Image Reconstructions

Fig. 9(a) shows multistatic synthetic aperture image reconstructions using multistatic data sets recovered from the walking aperture transmit sequence in Table III based on the Field II simulation. The results show that the adjoint decoding is unable to recover the imaging resolution achievable using the true FSA data set. Compared with the ground-truth multistatic synthetic aperture image reconstruction, the relative root-mean square error in image reconstruction is 2.01% when using the regularized inversion as opposed to 59.29% when using the adjoint. The speckle structure of the image reconstructed based on the data set recovered using the Tikhonov regularization better matches the image reconstructed using the true multistatic data set. Regularized inversion achieves the same contrast of 35 dB achieved by the true multistatic data set for the anechoic lesion, which is higher than the contrast of 25 dB observed when using the adjoint. However, when simulating the L12-3v probe, Fig. 9(b) shows that although regularized inversion results in multistatic images that agree more strongly with the ground truth (46.86% versus 9.63%

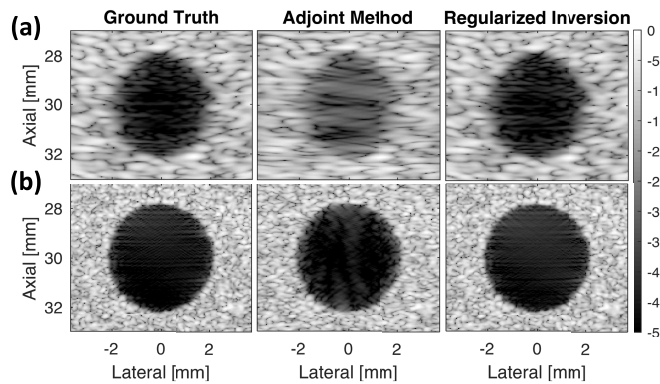


**Fig. 7.** Correlation between the true and recovered multistatic data set as a function of the angular spacing and extent of the phased-array sequence (see Table II) for the adjoint, the ramp-filtered adjoint, and the Tikhonov regularized inversion with  $\gamma = 0.1$ . (a) Total angular extent was kept at  $90^\circ$ . The correlation between the true and recovered data set starts to dropoff between  $1.0^\circ$  and  $1.5^\circ$  spacing. (b) Angular spacing between beams was fixed to  $0.5^\circ$ . The correlation between the true and recovered data set gradually increases with the total angular extent.

relative root mean square error), the adjoint still yields a well-resolved image because of the large transmit and receive

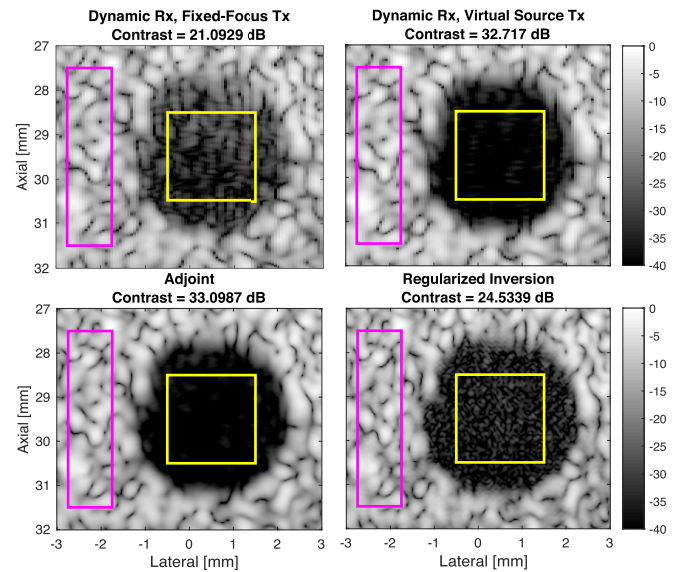


**Fig. 8.** Correlation between the true and recovered multistatic data set as a function of the lateral extent of the walking aperture sequence (see Table III) for the adjoint, the ramp-filtered adjoint, and the Tikhonov regularized inversion with  $\gamma = 0.01$ . (a) Maximum lateral extent was kept at 7.3 mm. The correlation between the true and recovered data set starts to drop off quickly between the 0.1- and 0.15-mm spacings. (b) Lateral spacing between beam origins was fixed to 0.05 mm. The correlation between the true and recovered data set gradually increases with the total lateral extent.



**Fig. 9.** Multistatic synthetic aperture image reconstruction based on data sets recovered from using the adjoint and the Tikhonov-regularized inversion ( $\gamma = 0.01$ ) to decode multistatic data sets from the walking aperture sequences shown in Tables III and IV compared to ground-truth multistatic data sets simulated in Field II. (a) In the top row of images, there is a 59.29% rms error between the adjoint and the ground truth, while there is only a 2.01% rms error between regularized inversion and the ground-truth images. (b) In the bottom row of images, there is a 46.86% rms error between the adjoint and the ground truth, while there is only a 9.63% rms error between the regularized inversion and the ground-truth images.

aperture after recovering the multistatic data set. The correlation between the ground truth and the recovered data set is



**Fig. 10.** Image reconstructions of the anechoic lesion using dynamic-receive focusing, virtual source synthetic aperture, and multistatic synthetic aperture using the adjoint and the Tikhonov-regularized inverse ( $\gamma = 0.01$ ) to decode data captured from the walking aperture transmit sequence on the Verasonics L12-3v linear array. The contrast between the lesion (yellow box) and background (magenta box) improves when going from dynamic-receive focusing to virtual source synthetic aperture to multistatic synthetic aperture with the adjoint decoding. Regularized inversion decreases lesion contrast due to noise amplification, but the visibility of the edge of the lesion still improves.

0.70355 for the adjoint method and 0.91156 for regularized inversion.

Fig. 10 shows image reconstructions of an anechoic lesion from the ATS Model 549 phantom by dynamic-receive focusing a single A-line for each transmit beam, virtual source synthetic aperture, and multistatic synthetic aperture using the adjoint and regularized inversion for data set recovery. The lesion contrast steadily improves when going from the dynamic-receive focusing each separate A-lines to virtual source synthetic aperture and the multistatic reconstruction based on the adjoint recovered data set. However, a drop in lesion contrast is observed when using regularized inversion for data set recovery, mainly due to the significant presence of electronic noise in the channel data.

Fig. 11 shows point target reconstructions from the ATS Model 549 phantom using the same four imaging techniques used in Fig. 10. The point target resolution greatly improves when going from dynamic-receive focusing each individual A-lines (0.321 mm) and virtual source synthetic aperture (0.333 mm) to multistatic synthetic aperture based on data sets recovered by the adjoint (0.252 mm) and regularized inversion (0.204 mm).

Fig. 12 show *in vivo* images of the rat liver and spine for each of the four imaging techniques used in Figs. 10 and 11. The interface between the spine and the back of the liver is enlarged to show differences between the four imaging techniques. Dynamic-receive focusing each A-line greatly diminishes tissue contrast between the liver and the anechoic region adjacent to the spine and reduces the visibility of the interface. Virtual source synthetic aperture improves this

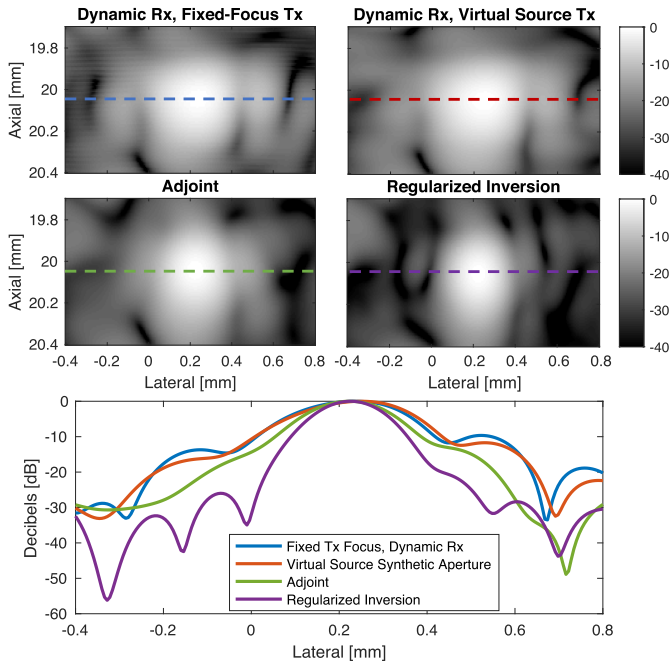


Fig. 11. Image reconstructions of point target using dynamic-receive focusing, virtual source synthetic aperture, and multistatic synthetic aperture with the adjoint and the Tikhonov-regularized inverse ( $\gamma = 0.01$ ) to decode data captured using a walking aperture transmit sequence on the Verasonics L12-3v linear array. The FWHM point target resolution for each technique is 0.321, 0.333, 0.252, and 0.204 mm, respectively.

contrast and makes this boundary clearly visible. Similar improvements are observed in the multistatic synthetic aperture image reconstruction when using the adjoint method to recover the multistatic channel data. Regularized inversion further sharpens the interface between the liver and spine.

### E. Coherence of Recovered Multistatic Data Sets

When multistatic channel data are transmit-focused in the absence of noise, reverberation, and phase aberration, the van-Cittert Zernike (VCZ) theorem [1] predicts that the coherence of speckle measured as a function of spatial lag, or offset between receive elements, should decay linearly from 1 to 0 starting from a lag of 0 to a lag of  $N_{\text{elem}}$ . Fig. 13(a) shows that when the adjoint is used to recover the multistatic data set from walking aperture channel data, the measured coherence decays more quickly as a function of lag than as predicted by the VCZ theorem. Because regularized inversion better recovers the single-element contributions to multistatic data set from walking aperture channel data, the coherence shown in 13(b) produces the results expected by the VCZ theorem.

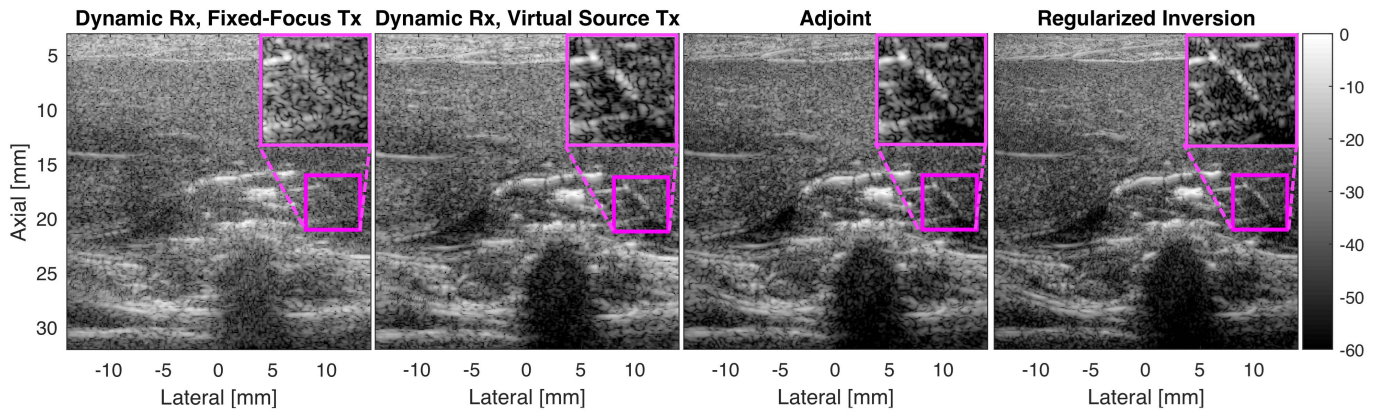
Fig. 13 uses Field II simulations to show that decoding via regularized inversion better retains the coherence structure expected in multistatic data in the absence of noise, reverberation, and phase aberration. However, channel data captured from an ultrasound scanner may contain one or more of these nonidealities. The channel data used in Fig. 14 were acquired by a Verasonics L12-3v on an ATS Model 549 phantom. This channel data have low SNR due to the highly attenuating medium of the phantom and the high frequency of the probe. Fig. 14 shows that coherence decays the quickest as a function

of receiver lag when the adjoint is used to recover the multistatic data set. The adjoint decoding can be thought of as a special case of regularized inversion, where  $\gamma = \infty$ . As the regularization  $\gamma$  decreases, the coherence becomes more triangular in shape; however, the lag-one coherence also decreases because the amount of noise present in the recovered signal also increases.

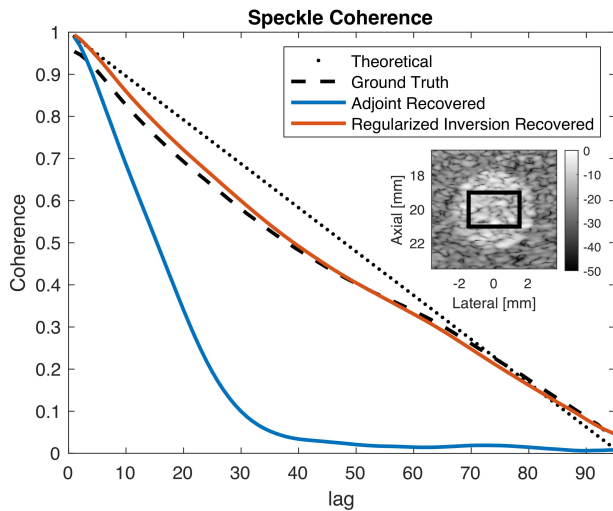
## V. DISCUSSION

The goal of REFoCUS is to accurately recover the multistatic synthetic aperture data set from channel data acquired using a wide variety of ultrasound sequences currently employed on conventional ultrasound scanners. This scheme would effectively standardize the data set and method used to perform image reconstruction. Furthermore, ultrasound image reconstruction is no longer limited by sound speed errors and phase aberration due to the fixed-focusing of transmit beams. Because the individual element contributions have been isolated, the sound speed used for the complete two-way focusing in multistatic synthetic aperture can be adjusted regardless of the assumed sound speed during the initial data acquisition. For example, although the delays for the transmit beams used on the Verasonics L12-3v were based on a sound speed of 1540 m/s, the recovered multistatic data sets for the ATS Model 549 phantom were focused at 1460 m/s for multistatic synthetic aperture image reconstruction. Furthermore, phase aberration correction can be performed over both transmit and receive apertures leading to optimal two-way focusing [4]. Accurate estimation and correction of phase aberration can also be used to determine the sound speed in the medium [3], [4].

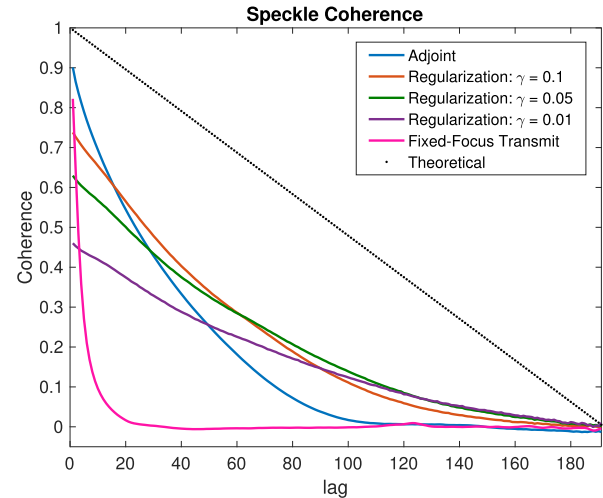
REFoCUS was initially demonstrated on phased arrays with a steered-transmit sequence using the adjoint method [22]. Fig. 2 reveals two key issues with this initial approach. The first issue is that the passband gain at each temporal frequency  $f$  should ideally be unity. Instead, the passband gain varies inversely proportional to the temporal frequency  $f$ , as predicted by (14). The ideal correction for this temporal frequency-dependent passband gain is to apply the ramp filter (15) that cancels it. After ramp filtering, the adjoint method works well at decoding the multistatic data set from the steered-transmit sequence mainly because of the flatness of the passbands shown in Fig. 2. The behavior of the adjoint in this application is very similar to how a matched filter, which is also an adjoint, deconvolves a chirp with a flat magnitude response [27]. The second issue is that at each temporal frequency  $f$ , the coverage of lateral spatial frequencies is limited to a passband whose cutoff is given by  $|k_x| < k_{x,\text{max}}(f) = (N_{\text{beams}} f \Delta \theta) / (2c)$ . This cutoff is a result of the limited angular coverage of transmit beams in the steered-transmit sequence. Just as in the case of the chirp, where the range of frequencies swept by the chirp affects how well the matched filter deconvolves the data, the performance of the adjoint method is limited by the total angle swept by transmit beams, as previously shown in [22]. Fig. 7(a) shows that channel data recovery gradually improves with the angular extent of the phased array sequence for each pseudoinverse.



**Fig. 12.** Image reconstructions of rat liver and spine using dynamic-receive focusing, virtual source synthetic aperture, and multistatic synthetic aperture using the adjoint and the Tikhonov-regularized inverse ( $\gamma = 0.01$ ) to decode data captured from the walking aperture transmit sequence on the Verasonics L12-3v linear array. The visibility and sharpness of the interface between the liver and spine (inset) of the rat improve with each successive image.



**Fig. 13.** Spatial coherence as a result of using the adjoint and the Tikhonov-regularized inverse ( $\gamma = 0.01$ ) to decode the multistatic data set from walking aperture sequences simulated in Field II. The spatial coherence of recovered multistatic channel data is compared with the spatial coherence of the ground truth and the theoretical coherence based on the VCZ theorem [1]. Regularized inversion results in coherence behavior that agrees well with ground-truth multistatic channel data, while the adjoint results in coherence that decays linearly toward zero over a small number of lags.



**Fig. 14.** Multistatic synthetic aperture image reconstruction using the adjoint and the Tikhonov-regularized inverse ( $\gamma = 0.1, 0.05$ , and  $0.01$ ) to decode the multistatic data set from the walking aperture channel data captured on the Verasonics L12-3v linear array. Spatial coherence was measured over a region of speckle ranging from  $-10$  to  $10$  mm laterally and  $20$  to  $40$  mm in depth. The adjoint results in spatial coherence that decays quickly as a function of lag. Regularized inversion results in a coherence that is more triangular in shape and consistent with the VCZ theorem, but the lag-one coherence drops as the regularization parameter  $\gamma$  decreases.

In addition to the total angle spanned, the angular spacing between the beams also plays an important role in channel data recovery. Based on the passband spatial frequency cutoff  $k_{x,\max}(f) = (N_{\text{beams}} f \Delta \theta) / (2c) \geq (1) / (\Delta x)$ , the Nyquist angular sampling for channel data recovery is  $\Delta \theta \leq (2c) / (N_{\text{beams}} f \Delta x) = (2(1540(\text{m}) / (\text{s}))) / ((181)(5 \times 10^6 \text{ Hz})(0.154 \times 10^{-3} \text{ m})) = 0.0221 \text{ rad} = 1.2662^\circ$ . Fig. 7(b) shows the first dropoff in the correlation between the recovered and ground-truth channel data when the angular spacing increases from  $1.0^\circ$  to  $1.5^\circ$ . This dropoff appears to correspond to the Nyquist angular sampling.

The role of the adjoint method in recovering the multistatic data set from the steered-transmit sequence is similar to the role that backprojection plays in CT reconstruction. Each beam in the steered-transmit sequence roughly samples a radial line along k-space at an angle equal to the steering angle. This is

much like the k-space sampling in parallel-beam CT, where each projection at a different angle corresponds to a different radial line in k-space by the Fourier Slice theorem [25], [26]. In both cases, the adjoint process is a backprojection into k-space along the same lines over which the information was sampled. As a result, the adjoint overemphasizes content at the center of k-space because each radial slice intersects there. In both cases, the compensating correction is a ramp filter. Similar corrections to adjoints are also seen in geophysics [27].

The main advantages of applying adjoints to inversion problems are that they are often easy to apply and they maximize SNR. It can be shown that adjoints maximize SNR by showing that as  $\gamma$  approach  $\infty$ , the Tikhonov-regularized inverse becomes the adjoint. Since  $\gamma$  is a penalty on the magnitude of the recovered frequency-domain signal and is intended to suppress noise, the adjoint maximally suppresses

noise. However, if the adjoint does not accurately model inversion, then although SNR is maximized, the signal recovered by the adjoint may not accurately solve the recovery problem. For the walking-aperture sequence, the singular values of  $\mathbf{H}(f)$  no longer show a flat passband that is clearly identifiable (see Fig. 4). This nonuniformity in singular values, both as a function of component number and frequency, shows that inversion is no longer accurately achieved using an adjoint. As a result, when the adjoint is applied to the walking aperture transmit sequence, the imaging resolution is suboptimal (see Figs. 9 and 11). Although the improvement in channel data recovery when moving from the adjoint to regularized inversion is similar for both array geometries [see Fig. 9(a) and (b)], the correlation of 0.91156 between the ground-truth multistatic channel data and the data set recovered by regularized inversion for Fig. 9(b) is significantly lower than the 0.9936 correlation observed in Fig. 6(f). Due to the large pitch and high frequency of the L12-3v array, aliasing-related errors due to the lateral sampling of the array may explain why the multistatic data set is not recovered as accurately as the case shown in Fig. 6(f), where the pitch is small and the beams are densely sampled.

The goal of regularized inversion is to more accurately recover the multistatic data set. Although low values of  $\gamma$  yield pseudoinverses that best achieve inversion of the forward model matrix, these same pseudoinverses can amplify noise present in the walking aperture channel data directly acquired by the system if  $\gamma$  is too low. Fig. 10 shows a drop in lesion contrast despite sharper lesion boundaries when comparing the results of regularized inversion to the adjoint. Fig. 14 shows that as  $\gamma$  decreases, although the coherence curve becomes more triangular in shape, the lag-one coherence also decreases. The decrease in lag-one coherence shows that the contribution of noise to recovered multistatic channel data increases as  $\gamma$  decreases. However, the remaining nonnoise component of the recovered multistatic channel data has a spatial coherence curve that becomes increasingly triangular as  $\gamma$  decreases.

Recovery of the multistatic data set is also affected by the beam density and lateral extent of the walking aperture sequence. Fig. 8(a) shows that channel data recovery gradually improves with the total lateral extent of the walking aperture sequence. Fig. 8(b) shows that for each pseudoinverse, the correlation between the recovered and ground-truth channel data first drops significantly when the spacing between beams increases from 0.1 to 0.15 mm. This implies that the beams in the walking aperture sequence are oversampled by at least a factor of 2. The walking aperture sequence shown in Table III was designed with densely spaced beams that approximately allow for the full 8-mm transmit aperture at the edges in order to create a challenging case where each beam provides little additional information about each transmit element on the array. Based on the linear systems formulation (2) of the recovery problem, the contribution of edge elements is easier to isolate as the beams in the walking aperture sequence are walked past the edges of the array. This would continue to hold as the beam origin is walked past the edges of the array. Furthermore, the dense spacing between beams would yield little new information about each element on the array.

This work investigates multistatic data set recovery with phased array and linear scan geometries; however, the inversions discussed here are not limited solely to steered and translated sequences and can be applied more broadly to sequences, such as vector and steered linear scans as well as to curvilinear arrays. When applying regularized inversion to recover the multistatic data set, the amount of regularization needed depends on the application. For example, the B-mode images reconstructed from the multistatic data set can tolerate a large amount of noise because summation across all transmitter-receiver pairs suppresses noise. Thus, for multistatic synthetic aperture image reconstruction,  $\gamma$  can be low. However, applications such as phase aberration correction require reliable estimates of cross correlation between individual receive channel signals in order to obtain accurate delays between receive channels. These cross correlations require higher SNR from the recovered multistatic data set and, hence, larger  $\gamma$ . The main limitation of regularized inversion is the tradeoff between SNR and the accuracy of the recovered signal. The choice of  $\gamma$  depends on both the end application and the nature of this tradeoff for the specific transmit sequence used. Future work should consider methodologies that break this tradeoff.

Sample code and data have been made available at <https://github.com/nbottenus/REFoCUS> (DOI: <https://zenodo.org/record/3473561>). The sample code provides example scripts for recovery of the multistatic data set using each pseudoinverse: the adjoint, the ramp-filtered adjoint, and the Tikhonov regularized inversion. We provide a Field II simulated multistatic data set from which focused transmit data sets are generated and a walking-aperture focused-transmit data set acquired from an ATS Model 549 phantom using an L12-3v probe on a Verasonics scanner. Finally, we also provide code that shows the effects of each pseudoinverse on the recovery of the k-space and wavefield for each single-element transmission.

## VI. CONCLUSION

For phased-array transmit sequences, a ramp-filter correction to the adjoint-based decoding yields significant improvements in multistatic synthetic aperture data set recovery. The Tikhonov regularization yields further improvements in multistatic data set recovery. For walking-aperture transmit sequences, the Tikhonov regularization yields highly accurate decoding at the cost of SNR. Future work should continue to examine the impact of the k-space response and SNR of these encoding-and-decoding schemes on end applications, such as coherence imaging, sound speed estimation, and phase aberration correction.

## APPENDIX TRANSMIT K-SPACE CALCULATION

The wave equation given by

$$\nabla^2 \Phi(\vec{x}, t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \Phi(\vec{x}, t) = S(x, t) \quad (28)$$

is used to derive the k-space response of a focused transmit beam. The k-space response here refers to the spatial Fourier

transform of the wavefield emitted into the medium (such as in [33] and [34]) as opposed to the pulse-echo response [35] or the point spread function [36]. The source term  $S(x, t)$  given by

$$S(x, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(\vec{x}_0, f) P(f) e^{j2\pi f t} \delta(\vec{x} - \vec{x}_0) d\vec{x}_0 df \quad (29)$$

describes a focused transmit pulse, where  $P(f)$  is the pulse spectrum as a function of frequency  $f$ ,  $\vec{x}_0$  is the transducer element coordinate, and  $a(\vec{x}_0, f) = a(\vec{x}_0) e^{-j2\pi f \tau(\vec{x}_0)}$  is the complex aperture function, where  $a(\vec{x}_0)$  and  $\tau(\vec{x}_0)$  are the apodization and transmit delays as a function of  $\vec{x}_0$ . The k-space solution to (28) with source term (29) is

$$\Phi(\vec{k}, t) = \frac{j e^{j2\pi c |\vec{k}| t}}{4\pi |\vec{k}|} \int_{-\infty}^{\infty} a(\vec{x}_0, c|\vec{k}|) P(c|\vec{k}|) e^{-j2\pi (\vec{k} \cdot \vec{x}_0)} d\vec{x}_0. \quad (30)$$

The k-space solution (30) assumes that each transducer element is a spherically emitting monopole source. When refined to include the  $\cos\theta$  dipole directivity of each transducer element, the solution is given as

$$\hat{\Phi}(\vec{k}, t) = \Phi(\vec{k}, t) \cos\left(\arctan \frac{k_x}{k_z}\right). \quad (31)$$

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